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A.A. Darwish

M.A. Seddeek

Reda Gamal Ahmed

Ahmed A. El-Deeb

Moataz Ashraf Mohamed

See next page for additional authors

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ORIGINAL STUDY

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Adel Abd El-Fattah Darwish [a], Mohamed Abd El-Hamed Seddeek [a], Reda Gamal Ahmed [b], Ahmed A. El-Deeb [b], Moataz Ashraf Mohamed [c,*], Hossam Hasan Abd El-Ghany [c]

[a] Department of Mathematics, Faculty of Science, Helwan University, Cairo, 11795, Egypt
[b] Department of Mathematics, Faculty of Science, Al-Azhar University, Cairo, 11651, Egypt
[c] Department of Mathematics, Faculty of Technology and Education, Helwan University, Cairo, 11795, Egypt

Abstract

We present in this paper a new type of self-reference functional integro-differential equation with a nonlocal initial condition. Also, we present a nonlocal problem of the functional integro-differential equation with the infinite point boundary condition for more generalization. An integral representation equivalent to the functional integro-differential equation is obtained to use the theorems needed for proving the existence, and uniqueness. Then we prove the continuous dependence of the solution on the nonlocal parameter and the initial data of the equation. To prove the existence of the solution of the equation we present the Schauder fixed point theorem for both finite and infinite boundary conditions, and to prove that this solution is unique we use the Banach fixed point theorem. At last, we produce an example of a self-reference functional integro-differential equation with a nonlocal initial condition to discuss the solution of that equation.

Keywords: Functional equations, Existence of solutions, Continuous dependence, State−dependence, Self-reference

1. Introduction

The nonlocal problem of functional differential equations has been studied by several researchers.

See for example Zhong and Zhang (2016); Srivastava et al. (2018); Zhang et al. (2018); El-Sayed and Ahmed (2020). Also, several studies devoted to such differential equations have lately been published.

For instance Kolomogorov et al. (1975); Eder (1984); Goebel and Kirk (1990); Wang (1990); Feckan (1993); Buica (1995); Stanek (1997); Stanek (2002); Berinde (2010); Zhang and Gong (2014); Darwish and Araz (2015).

In El-Deeb et al. (2019); El-Deeb et al. (2020a,b,c); Ali et al. (2023), the authors explored integral inequalities, which offer explicit bounds on unknown functions, have proven to be valuable in exploring the qualitative properties of solutions in differential, integral, and integro-differential equations.

This paper is devoted to proving the existence and uniqueness of the nonlocal problem of functional integro-differential equation in the form

$$\begin{cases} \dfrac{d\zeta}{d\eta}(\eta) = \varphi\left(\eta, \zeta\left(\displaystyle\int_0^{\phi_1(\eta)} h_1(\omega, \zeta(\omega))d\omega\right),\right. \\ \left.\zeta\left(\displaystyle\int_0^{\phi_2(\eta)} h_2(\omega, \zeta(\omega))d\omega\right)\right), \quad \eta \in I = (0, T] \\ \zeta(0) + \displaystyle\sum_{k=1}^{m} a_k \zeta(\tau_k) = \zeta_0, \quad \tau_k \in I. \end{cases} \tag{1.1}$$

The existence of continuous solution $\zeta \in C[0, T]$ under assumptions on the functional φ, h_1 and h_2 is studied. The uniqueness of the solution can be

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* Corresponding author.
E-mail addresses: profdarwish@yahoo.com (A.A.E.-F. Darwish), seddeek_m@hotmail.com (M.A.E.-H. Seddeek), redagamal@azhar.edu.eg (R.G. Ahmed), ahmedeldeeb@azhar.edu.eg (A.A. El-Deeb), moataz.ashraf@techedu.helwan.edu.eg (M.A. Mohamed), h.abdelghany@yahoo.com (H.H.A. El-Ghany).

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deduced when φ, h_1 and h_2 satisfy the Lipschitz condition. The continuous dependence of the unique solution on both the nonlocal parameter a_k and the initial data ζ_0 is proved.

Moreover, as an application, we study the nonlocal problem of functional integro-differential equation with the infinite-point boundary condition

$$\begin{cases} \frac{d\zeta}{d\eta}(\eta) = \varphi\left(\eta, \zeta\left(\int_0^{\phi_1(\eta)} h_1(\omega, \zeta(\omega))d\omega\right), \zeta\left(\int_0^{\phi_2(\eta)} h_2(\omega, \zeta(\omega))d\omega\right)\right), \quad \eta \in I \\ \zeta(0) + \sum_{k=1}^{\infty} a_k \zeta(\tau_k) = \zeta_0, \quad \tau_k \in I, \text{ if } \sum_{k=1}^{\infty} a_k \text{ is convergent.} \end{cases} \tag{1.2}$$

Here are some assumptions which, we will use later in our main result:

($\mathbb{H}_1$) $\varphi : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Carathéodory condition. There exists a function $c(\eta) \in L_1[0, T]$ is bounded and $b > 0$, such that

$$|\varphi(\eta, \alpha, \beta)| \le c(\eta) + b|\alpha| + b|\beta|, \quad |c(\eta)| \le M,$$

($\mathbb{H}_2$) $h_i : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition, such that

$$|h_i(\eta, \alpha)| \le 1, \quad \forall i = 1, 2,$$

($\mathbb{H}_3$) $2bT(1 - \Upsilon^{-1}) < 1$, where $\Upsilon^{-1} = \frac{1}{1+\sum_{k=1}^{m} a_k}$, $a_k > 0$,

($\mathbb{H}_4$) ϕ_i: $[0, T] \to [0, T]$ are continuous and nondecreasing, $\forall i = 1, 2$,

($\mathbb{H}_5$) $\varphi : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is measurable in η and satisfies the Lipschitz condition

$$|\varphi(\eta, u_1, u_2) - \varphi(\eta, v_1, v_2)| \le b \sum_{i=1}^{2} |u_i - v_i|, \quad \forall u_i, v_i \in \mathbb{R},$$

($\mathbb{H}_6$) $h_i : [0, T] \times \mathbb{R} \to \mathbb{R}$ is measurable in η and satisfies the Lipschitz condition

$$|h_i(\eta, \alpha) - h_i(\eta, u)| \le b_i|\alpha - u|, \quad \forall b_i > 0, \quad i = 1, 2, \quad \alpha, u \in \mathbb{R},$$

($\mathbb{H}_7$) $\lambda < 1$, where $\lambda = 2bLT^2(b_1 + b_2)(2 - \Upsilon^{-1}) + 4bT$,

($\mathbb{H}_8$) For all $\zeta \in C[0, T]$, there exists $K > 0$ such that $\|\zeta\| \le K$,

($\mathbb{H}_9$) $\int_0^T |h_i(\vartheta, 0)| d\vartheta = M_1$, $M_1 > 0$, $\forall i = 1, 2$.

2. Methods

This section is devoted to presenting some basic definitions and theorems which are needed in our main results.

Definition 2.1. (See Kolomogorov et al. (1975)). A set $M \subset L$ is said to be convex if whenever it contains two points ζ and ξ, it also contains the segment joining ζ and ξ.

Definition 2.2. (See Kolomogorov et al. (1975)). A family Φ of functions ϕ defined on a closed interval $[a, b]$ is said to be uniformly bounded if there exists a number $K > 0$ such that

$$|\phi(\zeta)| \le K$$

for all $\zeta \in [a, b]$ and all $\phi \in \Phi$.

Definition 2.3. (See Kolomogorov et al. (1975)). A family Φ of functions ϕ defined on a closed interval $[a, b]$ is said to be equicontinuous if given any $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|\zeta' - \zeta''| < \delta$ implies

$$|\phi(\zeta') - \phi(\zeta'')| < \varepsilon$$

for all $\zeta', \zeta'' \in [a, b]$ and all $\phi \in \Phi$.

Theorem 2.1. *[Kolomogorov et al., 1975, Arzelà] A necessary and sufficient condition for a family Φ of continuous functions ϕ defined on a closed interval $[a, b]$ to be relatively compact in $C[a, b]$ is that Φ be uniformly bounded and equicontinuous.*

Theorem 2.2. *[Kolomogorov et al., 1975, Lebesgue's Bounded Convergence] Let $\{f_n\}$ be a sequence of functions converging to a limit f on A, and suppos*

$$|f_n(\zeta)| \le \phi(\zeta) \quad (\zeta \in A, n = 1, 2, \ldots),$$

where ϕ is integrable on A . Then f is integrable on A and

$$\lim_{n\to\infty}\int_A f_n(\zeta)d\mu = \int_A f(\zeta)d\mu.$$

Theorem 2.3. *[Kolomogorov et al., 1975, The Fixed Point] Let A be a mapping of a metric space $\mathbb{R}$ into itself. Then ζ is called a fixed point of A if $Ax=\zeta$, i.e., if A carries ζ into itself.*

Theorem 2.4. *[Goebel and Kirk, 1990, Schauder Fixed Point] Let U be a convex subset of a Banach space X, and T: U → U is compact, continuous map. Then T has at least one fixed point in U.*

Definition 2.4. T: $U \to U$ is called a contraction operator if there is a constant $K \in [0, 1)$ such that

$$|T(u_1)-T(u_2)| \le K|u_1-u_2|$$

for each $u_1, u_2 \in U$.

Theorem 2.5. *[Goebel and Kirk, 1990, Banach Fixed Point] Let U be a closed subset of a Banach space Υ and T: U → U be a contraction, then T has a unique fixed point.*

Definition 2.5. The solution $\zeta \in AC[0, T]$ of the nonlocal problem 1.1 depends continuously on a_k and ζ_0, if

$$\forall\varepsilon>0,\quad \exists\delta_1(\varepsilon),\delta_2(\varepsilon)\quad \omega.\eta,\quad |a_k-a_k^*|<\delta_1, |\zeta_0-\zeta_0^*|<\delta_2 \Rightarrow \|\zeta-\zeta^*\|<\varepsilon,$$

where ζ^* is the solution of the nonlocal problem

$$\begin{cases}\dfrac{d\zeta^*}{d\eta}(\eta)=\varphi\left(\eta,\zeta^*\left(\displaystyle\int_0^{\phi_1(\eta)} h_1(\omega,\zeta^*(\omega))d\omega\right),\zeta^*\left(\displaystyle\int_0^{\phi_2(\eta)} h_2(\omega,\zeta^*(\omega))d\omega\right)\right), \quad \eta\in(0,T]\\ \zeta(0)+\displaystyle\sum_{k=1}^m a_k^*\zeta^*(\tau_k)=\zeta_0^*, \quad \tau_k\in(0,T).\end{cases} \tag{2.1}$$

3. Results and discussion

The equivalence of (1.1) and an integral equation is given in the following lemma.

Lemma 3.1. *Let $\mathbb{H}_1-\mathbb{H}_4$ be satisfied. Then ζ solves (1.1) iff*

$$\zeta(\eta)=\Upsilon^{-1}\left[\zeta_0-\sum_{k=1}^m a_k\int_0^{\tau_k}\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right] + \int_0^{\eta}\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega \tag{3.1}$$

where $\Upsilon^{-1}=\frac{1}{1+\sum_{k=1}^m a_k}, a_k>0.$

Proof. Let ζ be a solution of the boundary value problem of functional differential equation (1.1), integrating both sides of (1.1) we obtain

$$\zeta(\eta)=\zeta(0)+\int_0^{\eta}\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega \tag{3.2}$$

Using the nonlocal condition (1.1), we obtain

$$\sum_{k=1}^m a_k x(\tau_k) = \zeta(0)\sum_{k=1}^m a_k + \sum_{k=1}^m a_k\int_0^{\tau_k}\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega=\zeta_0-\zeta(0)$$

then

$$\zeta(0)\left(1+\sum_{k=1}^m a_k\right)=\zeta_0 - \sum_{k=1}^m a_k\int_0^{\tau_k}\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega$$

then we obtain

$$\zeta(0)=\frac{1}{1+\sum_{k=1}^{m}a_k}\left[\zeta_0-\sum_{k=1}^{m}a_k\int_0^{\tau_k}\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right]$$

Then

$$\zeta(0)=\Upsilon^{-1}\left[\zeta_0-\sum_{k=1}^{m}a_k\int_0^{\tau_k}\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right] \tag{3.3}$$

using equations (3.2)-(3.3), we deduce

$$\begin{aligned}\zeta(\eta)=\Upsilon^{-1}&\left[\zeta_0-\sum_{k=1}^{m}a_k\int_0^{\tau_k}\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right]\\&+\int_0^{\eta}\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\end{aligned}$$

Also, differentiation of equation (3.1) we obtain

$$\begin{aligned}\frac{d\zeta}{d\eta}\;=\;&\frac{d}{d\eta}\left\{\Upsilon^{-1}\left[\zeta_0-\sum_{k=1}^{m}a_k\int_0^{\tau_k}\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right]\right\}\\&+\frac{d}{d\eta}\left\{\int_0^{\eta}\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right\}\\&=0+\frac{d}{d\eta}\left\{\int_0^{\eta}\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right\}\\&=\varphi\left(\eta,\zeta\left(\int_0^{\phi_1(\eta)}h_1(\omega,\zeta(\omega))d\omega\right),\zeta\left(\int_0^{\phi_2(\eta)}h_2(\omega,\zeta(\omega))d\omega\right)\right).\end{aligned}$$

Also from integral equation (3.1) we obtain

$$\zeta(\tau_k) = \Upsilon^{-1}\left[\zeta_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right]$$
$$+\int_0^{\tau_k} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega$$

and

$$\sum_{k=1}^{m} a_k x(\tau_k) = \Upsilon^{-1}\sum_{k=1}^{m} a_k\left[\zeta_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right)\right.\right.$$
$$\left.\left., \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right]$$
$$+\sum_{k=1}^{m} a_k\int_0^{\tau_k} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega$$

Then

$$\zeta(0) + \sum_{k=1}^{m} a_k x(\tau_k) = \Upsilon^{-1}\left[\zeta_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right)\right.\right.$$
$$\left.\left., \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right]$$
$$+\Upsilon^{-1}\sum_{k=1}^{m} a_k\left[\zeta_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right]$$
$$+\sum_{k=1}^{m} a_k\int_0^{\tau_k} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega$$
$$= \Upsilon^{-1}\left(1+\sum_{k=1}^{m} a_k\right)\left[\zeta_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right]$$
$$+\sum_{k=1}^{m} a_k\int_0^{\tau_k} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega$$
$$= \zeta_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega$$
$$+\sum_{k=1}^{m} a_k\int_0^{\tau_k} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega$$
$$= \zeta_0$$

Using Schauder Fixed Point, Theorem 2.4 to establish the existence of at least one solution of (1.1) is shown in the following theorem.

Theorem 3.2. *Let $\mathbb{H}_1$ - $\mathbb{H}_5$ be satisfied, then (1.1) has at least one solution.*

Proof. Define the operator Ψ associated with the integral equation (3.1) as

$$\Psi\zeta(\eta) = \Upsilon^{-1}\left[\zeta_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right]$$
$$+ \int_0^{\eta} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega$$

and define the set S_L by

$$S_L = \{\zeta \in \mathbb{R} : |\zeta(\eta) - \zeta(\omega)| \leq L|\eta - \omega| \quad \forall \eta, \omega \in [0,T]\},$$
$$L = \frac{M + 2bE^{-1}|\zeta_0|}{1 + 2bTE^{-1} - 4bT}$$

Then we have for $\zeta \in C[0, T]$

$$\begin{aligned}
\left|\Psi\zeta(\eta)\right| &= \Bigg|\Upsilon^{-1}\left[\zeta_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right] \\
&+ \int_0^{\eta} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\Bigg| \\
&\leq \Upsilon^{-1}|\zeta_0| + \Upsilon^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} \left|\varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)\right|d\omega \\
&+ \int_0^{\eta} \left|\varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)\right|d\omega \\
&\leq \Upsilon^{-1}|\zeta_0| + \Upsilon^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} c(\omega)d\omega + bE^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} \left|\zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right)\right|d\omega \\
&+ bE^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} \left|\zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right|d\omega + \int_0^{\eta} c(\omega)d\omega \\
&+ b\int_0^{\eta} \left|\zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right)\right|d\omega + b\int_0^{\eta} \left|\zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right|d\omega \\
&\leq \Upsilon^{-1}|\zeta_0| + \Upsilon^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} c(\omega)d\omega + bE^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} \left|\zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right) - \zeta(0)\right|d\omega \\
&+ bE^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} \left|\zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right) - \zeta(0)\right|d\omega + 2bE^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} |\zeta(0)|d\omega \\
&+ \int_0^{\eta} c(\omega)d\omega + b\int_0^{\eta} \left|\zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right) - \zeta(0)\right|d\omega \\
&+ b\int_0^{\eta} \left|\zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right) - \zeta(0)\right|d\omega + 2b\int_0^{\eta} |\zeta(0)|d\omega \\
&\leq \Upsilon^{-1}|\zeta_0| + MT\left(1 - \Upsilon^{-1}\right) + bLE^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} \left|\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right|d\omega \\
&+ bLE^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} \left|\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta|d\omega + 2bT|\zeta(0)\right|\left(1 - \Upsilon^{-1}\right) \\
&+ MT + bL\int_0^{\eta} \left|\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right|d\omega \\
&+ bL\int_0^{\eta} \left|\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right|d\omega + 2bT|\zeta(0)|
\end{aligned}$$

$$\begin{aligned}
&\leq \Upsilon^{-1}|\zeta_0| + MT\left(2-\Upsilon^{-1}\right) + bLE^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k}\int_0^{\phi_1(\omega)} |h_1(\vartheta,\zeta(\vartheta))|d\vartheta d\omega \\
&+ bLE^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k}\int_0^{\phi_2(\omega)} |h_2(\vartheta,\zeta(\vartheta))|d\vartheta d\omega + bL\int_0^{\eta}\int_0^{\phi_1(\omega)} |h_1(\vartheta,\zeta(\vartheta))|d\vartheta d\omega \\
&+ bL\int_0^{\eta}\int_0^{\phi_2(\omega)} |h_2(\vartheta,\zeta(\vartheta))|d\vartheta d\omega + 2bT|\zeta(0)|\left(2-\Upsilon^{-1}\right) \\
&\leq \Upsilon^{-1}|\zeta_0| + MT\left(2-\Upsilon^{-1}\right) + 2bLT^2\left(2-\Upsilon^{-1}\right) + 2bT|\zeta(0)|\left(2-\Upsilon^{-1}\right)
\end{aligned} \tag{3.4}$$

Using equation (3.3) we obtain

$$\begin{aligned}
|\zeta(0)| &= \left|\Upsilon^{-1}\left[\zeta_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)\right]d\omega\right| \\
&\leq \Upsilon^{-1}|\zeta_0| + \Upsilon^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k}\left|\varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)\right|d\omega \\
&\leq \Upsilon^{-1}|\zeta_0| + \Upsilon^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k}\left[c(\omega) + b\left|\zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right)\right| + b\left|\zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right|\right]d\omega \\
&\leq \Upsilon^{-1}|\zeta_0| + \Upsilon^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k}\left[c(\omega) + b\left|\zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right) - \zeta(0)\right|\right. \\
&\left. + b\left|\zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right) - \zeta(0)\right| + 2b|\zeta(0)|\right]d\omega \\
&\leq \Upsilon^{-1}|\zeta_0| + \Upsilon^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k}\left[c(\omega) + bL\left|\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right|\right. \\
&\left. + bL\left|\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right| + 2b|\zeta(0)|\right]d\omega \\
&\leq \Upsilon^{-1}|\zeta_0| + \Upsilon^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k}\left[c(\omega) + bL\int_0^{\phi_1(\omega)} |h_1(\vartheta,\zeta(\vartheta))|\,d\vartheta + bL\int_0^{\phi_2(\omega)} |h_2(\vartheta,\zeta(\vartheta))|\,d\vartheta\right. \\
&\left. + 2b|\zeta(0)|\right]d\omega \\
&\leq \Upsilon^{-1}|\zeta_0| + \left[MT + 2bLT^2 + 2bT|\zeta(0)|\right]\Upsilon^{-1}\sum_{k=1}^{m} a_k = \Upsilon^{-1}|\zeta_0| + \left[MT + 2bLT^2 + 2bT|\zeta(0)|\right]\left(1-\Upsilon^{-1}\right)
\end{aligned}$$

Then

$$|\zeta(0)| \leq \frac{\Upsilon^{-1}|\zeta_0| + MT + 2bLT^2 - MTE^{-1} - 2bLT^2\Upsilon^{-1}}{1 - 2bT + 2bTE^{-1}} = N. \tag{3.5}$$

From equations (3.4)–(3.5), we obtain

$$\begin{aligned} |\Psi\zeta(\eta)| &\leq \Upsilon^{-1}|\zeta_0| + (2 - \Upsilon^{-1})\left(MT + 2bLT^2 + 2bT*\frac{\Upsilon^{-1}|\zeta_0| + MT + 2bLT^2 - MTE^{-1} - 2bLT^2\Upsilon^{-1}}{1 - 2bT + 2bTE^{-1}}\right) \\ &\leq \Upsilon^{-1}|\zeta_0| + \left(\frac{MT + 2bLT^2 + 2bT|\zeta_0|\Upsilon^{-1}}{1 - 2bT + 2bTE^{-1}}\right)(2 - \Upsilon^{-1}) \\ &= \frac{\Upsilon^{-1}|\zeta_0| + MT - MTE^{-1} + 2bLT^2 - 2bLT^2\Upsilon^{-1}}{1 - 2bT + 2bTE^{-1}} + \left(\frac{M + 2bLT + 2bE^{-1}|\zeta_0|}{1 - 2bT + 2bTE^{-1}}\right)T \\ &= \frac{\Upsilon^{-1}|\zeta_0| + MT - MTE^{-1} + 2bLT^2 - 2bLT^2\Upsilon^{-1}}{1 - 2bT + 2bTE^{-1}} + LT \end{aligned}$$

Now let $\eta_1, \eta_2 \in (0, T]$, such that $|\eta_2 - \eta_1| < \delta$, then

$$\begin{aligned} &|\Psi\zeta(\eta_2) - \Psi\zeta(\eta_1)| \\ &= \left|\int_0^{\eta_2} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\right)\right)d\omega\right. \\ &\quad \left. - \int_0^{\eta_1} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\right)\right)d\omega\right| \\ &\leq \int_{\eta_1}^{\eta_2} \left|\varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\right)\right)\right|d\omega \\ &\leq \int_{\eta_1}^{\eta_2} \left[c(\omega) + b\left|\zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\right)\right| + b\left|\zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\right)\right|\right]d\omega \\ &\leq \int_{\eta_1}^{\eta_2} \left[c(\omega) + b\left|\zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\right) - \zeta(0)\right| + b\left|\zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\right) - \zeta(0)\right| + 2b\left|\zeta(0)\right|\right]d\omega \\ &\leq \int_{\eta_1}^{\eta_2} \left[c(\omega) + bL\int_0^{\phi_1(\omega)} |h_1(\vartheta, \zeta(\vartheta))|d\vartheta + bL\int_0^{\phi_2(\omega)} |h_2(\vartheta, \zeta(\vartheta))|d\vartheta + 2b|\zeta(0)|\right]d\omega \\ &\leq M(\eta_2 - \eta_1) + 2bLT(\eta_2 - \eta_1) + 2b|\zeta(0)|(\eta_2 - \eta_1) \\ &\leq M(\eta_2 - \eta_1) + 2bLT(\eta_2 - \eta_1) + 2b(\eta_2 - \eta_1)\left(\frac{\Upsilon^{-1}|\zeta_0| + MT + 2bLT^2 - MTE^{-1} - 2bLT^2\Upsilon^{-1}}{1 - 2bT + 2bTE^{-1}}\right) \\ &\leq (\eta_2 - \eta_1)\left(\frac{M + 2bLT + 2bE^{-1}|\zeta_0|}{1 + 2bTE^{-1} - 2bT}\right) \\ &\leq L\delta \end{aligned}$$

This proves that $\Psi: S_l \to S_L$, and the class $\{\Psi\zeta\}$ is uniformly bounded and equi-continous in S_L.

Let $\zeta_n \in S_L$, $\zeta_n \to \zeta (n \to \infty)$, then from continuity of φ and h_i, we obtain

$\varphi(\eta, \zeta_n(\eta), \xi_n(\eta)) \to \varphi(\eta, \zeta(\eta), \xi(\eta))$, and $h_i(\eta, \zeta_n(\eta)) \to h_i(\eta, \zeta(\eta))\ \forall i = 1, 2$.

Also

$$
\begin{aligned}
&|\Psi(\zeta_n)(\eta) - \Psi(\zeta)(\eta)| = |\Upsilon^{-1}\Bigg[\zeta_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \varphi\Bigg(\omega, \zeta_n\Bigg(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta_n(\vartheta))d\vartheta\Bigg)\\
&, \zeta_n\Bigg(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta_n(\vartheta))d\vartheta\Bigg)\Bigg)d\omega\Bigg]\\
&\quad + \int_0^{\eta} \varphi\Bigg(\omega, \zeta_n\Bigg(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta_n(\vartheta))d\vartheta\Bigg), \zeta_n\Bigg(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta_n(\vartheta))d\vartheta\Bigg)\Bigg)d\omega\\
&\quad -\Upsilon^{-1}\Bigg[\zeta_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \varphi\Bigg(\omega, \zeta\Bigg(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\Bigg), \zeta\Bigg(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\Bigg)\Bigg)d\omega\Bigg]\\
&\quad - \int_0^{\eta} \varphi\Bigg(\omega, \zeta\Bigg(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\Bigg), \zeta\Bigg(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\Bigg)\Bigg)d\omega\Bigg|\\
&\leq \Upsilon^{-1} \sum_{k=1}^{m} a_k \int_0^{\tau_k} \Bigg|\varphi\Bigg(\omega, \zeta\Bigg(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\Bigg), \zeta\Bigg(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\Bigg)\Bigg)\\
&\quad -\varphi\Bigg(\omega, \zeta_n\Bigg(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta_n(\vartheta))d\vartheta\Bigg), \zeta_n\Bigg(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta_n(\vartheta))d\vartheta\Bigg)\Bigg)\Bigg|d\omega\\
&\quad + \int_0^{\eta} \Bigg|\varphi\Bigg(\omega, \zeta_n\Bigg(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta_n(\vartheta))d\vartheta\Bigg), \zeta_n\Bigg(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta_n(\vartheta))d\vartheta\Bigg)\Bigg)\\
&\quad -\varphi\Bigg(\omega, \zeta\Bigg(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\Bigg), \zeta\Bigg(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\Bigg)\Bigg)\Bigg|\\
&\leq bE^{-1} \sum_{k=1}^{m} a_k \int_0^{\tau_k} \Bigg|\zeta\Bigg(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\Bigg) - \zeta_n\Bigg(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta_n(\vartheta))d\vartheta\Bigg)\Bigg|d\omega\\
&\quad +bE^{-1} \sum_{k=1}^{m} a_k \int_0^{\tau_k} \Bigg|\zeta\Bigg(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\Bigg) - \zeta_n\Bigg(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta_n(\vartheta))d\vartheta\Bigg)\Bigg|d\omega\\
&\quad +b \int_0^{\eta} \Bigg|\zeta_n\Bigg(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta_n(\vartheta))d\vartheta\Bigg) - \zeta\Bigg(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\Bigg)\Bigg|d\omega\\
&\quad +b \int_0^{\eta} \Bigg|\zeta_n\Bigg(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta_n(\vartheta))d\vartheta\Bigg) - \zeta\Bigg(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\Bigg)\Bigg|d\omega
\end{aligned}
$$

Now

$$\begin{aligned}&\left|\zeta_n\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta_n(\vartheta))d\vartheta\right)-\zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right)\right|\\&\le\left|\zeta_n\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta_n(\vartheta))d\vartheta\right)-\zeta_n\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right)\right|\\&\quad+\left|\zeta_n\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right)-\zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right)\right|\\&\le L\int_0^{\phi_1(\omega)}|h_1(\vartheta,\zeta_n(\vartheta))-h_1(\vartheta,\zeta(\vartheta))|d\vartheta+\frac{\varepsilon_1}{2}\\&\le\frac{\varepsilon_1}{2}+\frac{\varepsilon_1}{2}=\varepsilon_1.\end{aligned}$$

We can also deduce that

$$\begin{aligned}&\left|\zeta_n\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta_n(\vartheta))d\vartheta\right)\right.\\&\left.-\zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right|\\&\le\varepsilon_2.\end{aligned}$$

Then, we obtain

$$|\Psi(\zeta_n)(\eta)-\Psi(\zeta)(\eta)|\le bT\left(2-\Upsilon^{-1}\right)(\varepsilon_1+\varepsilon_2)=\varepsilon.$$

Then $\Psi\zeta_n \to \Psi\zeta$ as $n \to \infty$. This mean that the operator Ψ is continuous.

Hence by Schauder fixed point Theorem (Goebel and Kirk, 1990) there exist at least one solution $\zeta \in C[0, T]$ of the integral equation (3.1).

Theorem 3.3. *If $\mathbb{H}_1$ - $\mathbb{H}_4$ hold, then the nonlocal problem of (1.2) has at least one solution.*

Proof. Let $\sum_{k=1}^{m} a_k$ be convergent. Then

$$\begin{aligned}\zeta_m(\eta)=\Upsilon^{-1}&\left[\zeta_0-\sum_{k=1}^{m}a_k\int_0^{\tau_k}\varphi\left(\omega,\zeta_m\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta_m(\vartheta))d\vartheta\right),\zeta_m\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta_m(\vartheta))d\vartheta\right)\right)d\omega\right]\\&+\int_0^{\eta}\varphi\left(\omega,\zeta_m\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta_m(\vartheta))d\vartheta\right),\zeta_m\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta_m(\vartheta))d\vartheta\right)\right)d\omega.\end{aligned}\tag{3.6}$$

Take the limit to (3.6), as $m \to \infty$, we have

$$\begin{aligned}\lim_{m\to\infty}\zeta_m(\eta) = & \lim_{m\to\infty}\Upsilon^{-1}\left[\zeta_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \varphi\left(\omega, \zeta_m\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta_m(\vartheta))d\vartheta\right)\right.\right. \\ & \left.\left., \zeta_m\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta_m(\vartheta))d\vartheta\right)\right)d\omega\right] \\ & + \lim_{m\to\infty}\int_0^{\eta} \varphi\left(\omega, \zeta_m\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta_m(\vartheta))d\vartheta\right), \zeta_m\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta_m(\vartheta))d\vartheta\right)\right)d\omega.\end{aligned} \tag{3.7}$$

Now, $|a_k\zeta(\tau_k)| \le |a_k|\|\zeta\|$, then by comparison test $\sum_{k=1}^{\infty} a_k\zeta(\tau_k)$ is convergent. Moreover,

$$\begin{aligned}&\left|\int_0^{\tau_k} \varphi\left(\omega, \zeta_m\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta_m(\vartheta))d\vartheta\right), \zeta_m\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta_m(\vartheta))d\vartheta\right)\right)d\omega\right| \\ &\qquad \le MT + 2LbT^2 + 2bT|\zeta(0)| = M_2.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left|a_k \int_0^{\tau_k} \varphi\left(\omega, \zeta_m\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta_m(\vartheta))d\vartheta\right),\right.\right. \\ &\qquad \left.\left.\zeta_m\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta_m(\vartheta))d\vartheta\right)\right)d\omega\right| \\ &\qquad\qquad \le |a_k|M_2\end{aligned}$$

and by the comparison test

$$\begin{aligned}&\sum_{k=1}^{\infty} a_k \int_0^{\tau_k} \varphi\left(\omega, \zeta_m\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta_m(\vartheta))d\vartheta\right),\right. \\ &\qquad \left.\zeta_m\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta_m(\vartheta))d\vartheta\right)\right)d\omega,\end{aligned}$$

is convergent. Using assumptions $\mathbb{H}_1$ - $\mathbb{H}_2$ and Lebesgue's bounded convergence theorem (Kolomogorov et al., 1975), from (3.7), we obtain

$$\begin{aligned}\zeta(\eta) = & \frac{1}{1+\sum_{k=1}^{\infty} a_k}\left[\zeta_0 - \sum_{k=1}^{\infty} a_k \int_0^{\tau_k} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\right)\right)d\omega\right] \\ & + \int_0^{\eta} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\right)\right)d\omega.\end{aligned}$$

For the main theorems presented below, we employ the notation

$$\begin{aligned}\Xi &:= C(I, \mathbb{R}), \\ d(\zeta, \xi) &:= \sup_{\eta\in I}|\zeta(\eta) - \xi(\eta)| \quad for \quad \zeta, \xi \in \Xi, \\ \|\zeta\| &:= d(\zeta, 0) \quad for \quad \zeta \in \Xi,\end{aligned}$$

Using Banach Fixed Point, Theorem 2.5 to establish the existence of exactly one solution of (1.1) is shown in the following theorem.

Theorem 3.4. *Let $\mathbb{H}_4$- $\mathbb{H}_9$ be satisfied. Then the solution of (1.1) is unique.*

Proof. Firstly, we prove that the operator Ψ maps $C[0, T]$ to itself. Let $\zeta \in C[0, T]$ and $\eta_1, \eta_2 \in [0, T]$ such that $\eta_1 < \eta_2$ and $|\eta_2 - \eta_1| \le \delta$. Then

$$
\begin{aligned}
&|\Psi(\zeta)(\eta_2) - \Psi(\zeta)(\eta_1)| \\
&= |\Upsilon^{-1}\left[\zeta_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right] \\
&\quad + \int_0^{\eta_2} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega \\
&\quad -\Upsilon^{-1}\left[\zeta_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right] \\
&\quad - \int_0^{\eta_1} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\Bigg| \\
&\leq \int_{\eta_1}^{\eta_2} \left|\varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right| \\
&\leq \int_{\eta_1}^{\eta_2} \left[b\left|\zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right)\right| + b\left|\zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right| + |\varphi(\omega,0,0)|\right]d\omega \\
&\leq \int_{\eta_1}^{\eta_2} \left[b\left|\zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right) - \zeta(0) + \zeta(0)\right| + b\left|\zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right) - \zeta(0) + \zeta(0)\right|\right. \\
&\quad \left. + |\varphi(\omega,0,0)|\right]d\omega \\
&\leq \int_{\eta_1}^{\eta_2} \left[b\left|\zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right) - \zeta(0)\right| + b\left|\zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right) - \zeta(0)\right| + 2b|\zeta(0)|\right. \\
&\quad \left. + |\varphi(\omega,0,0)|\right]d\omega \\
&\leq \int_{\eta_1}^{\eta_2} \left[bL\left|\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\right| + bL\left|\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\right| + 2b|\zeta(0)| + |\varphi(\omega,0,0)|\right]d\omega \\
&\leq \int_{\eta_1}^{\eta_2} \left[bL\int_0^{\phi_1(\omega)} |h_1(\vartheta,\zeta(\vartheta))|\,d\vartheta + bL\int_0^{\phi_2(\omega)} |h_2(\vartheta,\zeta(\vartheta))|\,d\vartheta + 2b|\zeta(0)| + |\varphi(\omega,0,0)|\right]d\omega \\
&\leq \int_{\eta_1}^{\eta_2} \left[bL\int_0^{\phi_1(\omega)} (|h_1(\vartheta,0)| + b_1|\zeta(\vartheta)|)d\vartheta + bL\int_0^{\phi_2(\omega)} (|h_2(\vartheta,0)| + b_2|\zeta(\vartheta)|)d\vartheta\right. \\
&\quad \left. + 2b|\zeta(0)| + |\varphi(\omega,0,0)|\right]d\omega \\
&\quad \leq \int_{\eta_1}^{\eta_2} \left[bL\int_0^{\phi_1(\omega)} (|h_1(\vartheta,0)| + b_1\|\zeta\|)d\vartheta + bL\int_0^{\phi_2(\omega)} (|h_2(\vartheta,0)| + b_2\|\zeta\|)d\vartheta\right. \\
&\qquad\quad \left. + 2b|\zeta(0)| + |\varphi(\omega,0,0)|\right]d\omega \\
&\qquad \leq (\eta_2 - \eta_1)\left[|\varphi(\omega,0,0)| + 2bLM_1T + bb_1LKT^2 + bb_2LKT^2 + 2bN\right] \\
&\qquad\ \leq \delta\left[|\varphi(\omega,0,0)| + 2bLM_1T + bb_1LKT^2 + bb_2LKT^2 + 2bN\right] = \varepsilon
\end{aligned}
$$

This proves that $\Psi: C[0, T] \rightarrow C[0, T]$. Secondly, we prove that Ψ is contraction.

Let $\zeta, \xi \in C[0, T]$ Then

$$
\begin{aligned}
&|\Psi(\zeta)(\eta)-\Psi(\xi)(\eta)| \\
&= |\Upsilon^{-1}\left[\zeta_0-\sum_{k=1}^{m} a_k \int_0^{\tau_k} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\right)\right)d\omega\right] \\
&\quad + \int_0^{\eta} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\right)\right)d\omega \\
&\quad -\Upsilon^{-1}\left[\zeta_0-\sum_{k=1}^{m} a_k \int_0^{\tau_k} \varphi\left(\omega, \xi\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \xi(\vartheta))d\vartheta\right), \xi\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \xi(\vartheta))d\vartheta\right)\right)d\omega\right] \\
&\quad - \int_0^{\eta} \varphi\left(\omega, \xi\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \xi(\vartheta))d\vartheta\right), \xi\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \xi(\vartheta))d\vartheta\right)\right)d\omega \\
&\leq \Upsilon^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} \left|\varphi\left(\omega, \xi\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \xi(\vartheta))d\vartheta\right), \xi\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \xi(\vartheta))d\vartheta\right)\right)\right. \\
&\quad \left. -\varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\right)\right)\right| d\omega \\
&\quad + \int_0^{\eta} \left|\varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\right)\right)\right. \\
&\quad \left. -\varphi\left(\omega, \xi\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \xi(\vartheta))d\vartheta\right), \xi\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \xi(\vartheta))d\vartheta\right)\right)\right| d\omega \\
&\leq \Upsilon^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} \left[b\left|\xi\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \xi(\vartheta))d\vartheta\right) - \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\right)\right|\right. \\
&\quad \left. + b\left|\xi\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \xi(\vartheta))d\vartheta\right) - \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\right)\right|\right] d\omega \\
&\quad + \int_0^{\eta} \left[b\left|\zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\right) - \xi\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \xi(\vartheta))d\vartheta\right)\right|\right. \\
&\quad \left. + b\left|\zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\right) - \xi\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \xi(\vartheta))d\vartheta\right)\right|\right] d\omega
\end{aligned}
$$

$$
\begin{aligned}
&\leq \Upsilon^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k}\Bigg[b\bigg|\xi\bigg(\int_0^{\phi_1(\omega)} h_1(\vartheta,\xi(\vartheta))d\vartheta\bigg)-\xi\bigg(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\bigg)\bigg| \\
&\quad +b\bigg|\xi\bigg(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\bigg)-\zeta\bigg(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\bigg)\bigg| \\
&\quad +b\bigg|\xi\bigg(\int_0^{\phi_2(\omega)} h_2(\vartheta,\xi(\vartheta))d\vartheta\bigg)-\xi\bigg(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\bigg)\bigg| \\
&\quad +b\bigg|\xi\bigg(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\bigg)-\zeta\bigg(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\bigg)\bigg|\Bigg] \\
&\quad +\int_0^{\eta}\Bigg[b\bigg|\zeta\bigg(\int_0^{\phi_1(\omega)} h_1(\vartheta,\zeta(\vartheta))d\vartheta\bigg)-\zeta\bigg(\int_0^{\phi_1(\omega)} h_1(\vartheta,\xi(\vartheta))d\vartheta\bigg)\bigg| \\
&\quad +b\bigg|\zeta\bigg(\int_0^{\phi_1(\omega)} h_1(\vartheta,\xi(\vartheta))d\vartheta\bigg)-\xi\bigg(\int_0^{\phi_1(\omega)} h_1(\vartheta,\xi(\vartheta))d\vartheta\bigg)\bigg| \\
&\quad +b\bigg|\zeta\bigg(\int_0^{\phi_2(\omega)} h_2(\vartheta,\zeta(\vartheta))d\vartheta\bigg)-\zeta\bigg(\int_0^{\phi_2(\omega)} h_2(\vartheta,\xi(\vartheta))d\vartheta\bigg)\bigg| \\
&\quad +b\bigg|\zeta\bigg(\int_0^{\phi_2(\omega)} h_2(\vartheta,\xi(\vartheta))d\vartheta\bigg)-\xi\bigg(\int_0^{\phi_2(\omega)} h_2(\vartheta,\xi(\vartheta))d\vartheta\bigg)\bigg|\Bigg] \\
&\leq bLE^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k}\int_0^{\phi_1(\omega)} |h_1(\vartheta,\xi(\vartheta))-h_1(\vartheta,\zeta(\vartheta))|d\vartheta d\omega \\
&\quad +bLE^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k}\int_0^{\phi_2(\omega)} |h_2(\vartheta,\xi(\vartheta))-h_2(\vartheta,\zeta(\vartheta))|d\vartheta d\omega+2b\|\zeta-\xi\|T \\
&\quad +bL\int_0^{\eta}\int_0^{\phi_1(\omega)} |h_1(\vartheta,\zeta(\vartheta))-h_1(\vartheta,\xi(\vartheta))|d\vartheta d\omega+bL\int_0^{\eta}\int_0^{\phi_2(\omega)} |h_2(\vartheta,\zeta(\vartheta))-h_2(\vartheta,\xi(\vartheta))|d\vartheta d\omega \\
&\quad +2b\|\zeta-\xi\|T \\
&\leq bb_1LE^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k}\int_0^{\phi_1(\omega)} |\xi(\vartheta)-\zeta(\vartheta)|d\vartheta d\omega \\
&\quad +bb_2LE^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k}\int_0^{\phi_2(\omega)} |\xi(\vartheta)-\zeta(\vartheta)|d\vartheta d\omega \\
&\quad +bb_1L\int_0^{\eta}\int_0^{\phi_1(\omega)} |\xi(\vartheta)-\zeta(\vartheta)|d\vartheta d\omega+bb_2L\int_0^{\eta}\int_0^{\phi_2(\omega)} |\xi(\vartheta)-\zeta(\vartheta)|d\vartheta d\omega+4bT\|\zeta-\xi\| \\
&\leq 2bb_1LT^2\|\zeta-\xi\|\big(2-\Upsilon^{-1}\big)+2bb_2LT^2\|\zeta-\xi\|\big(2-\Upsilon^{-1}\big)+4bT\|\zeta-\xi\| \\
&= \big(2bLT^2(b_1+b_2)\big(2-\Upsilon^{-1}\big)+4bT\big)\|\zeta-\xi\| \\
&= \lambda d(\zeta,\xi)
\end{aligned}
$$

Since $\lambda < 1$, then Ψ is contraction operator.

Then by Banach Fixed Point Theorem (Goebel and Kirk, 1990), the solution $\zeta \in C[0, T]$ is unique.

Remark 3.1. Under the conditions of Theorem (3.4), if ζ_i, $i \in \mathbb{N}$, are defined recursively by

$$\zeta_{i+1} = \Upsilon^{-1}\left[\zeta_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\right)\right)d\omega\right] + \int_0^{\eta} \varphi\left(\omega, \zeta\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta(\vartheta))d\vartheta\right), \zeta\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta(\vartheta))d\vartheta\right)\right)d\omega$$

then $\zeta_i \to \zeta^*$ uniformly on I, where ζ^* is the only solution of (1.1)

Using the assumptions of Theorem (3.4) to establish the continuous dependence of both the nonlocal parameter a_k and the initial data ζ_0 is shown in the following theorem.

Theorem 3.5. *Let the assumptions of Theorem (3.4) be satisfied, and* $\sup_{\eta \in [0,T]} \int_0^T \varphi(\omega, 0, 0) = M_3$ *then the solution of (1.1) depends continuously on both the nonlocal parameter* a_k *and the initial data* ζ_0*.*

Proof. Let ζ^* is the solution of the integral equation

$$\zeta^*(\eta) = \Upsilon^{*-1}\left[\zeta_0^* - \sum_{k=1}^{m} a_k^* \int_0^{\tau_k} \varphi\left(\omega, \zeta^*\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta^*(\vartheta))d\vartheta\right), \zeta^*\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta^*(\vartheta))d\vartheta\right)\right)d\omega\right] + \int_0^{\eta} \varphi\left(\omega, \zeta^*\left(\int_0^{\phi_1(\omega)} h_1(\vartheta, \zeta^*(\vartheta))d\vartheta\right), \zeta^*\left(\int_0^{\phi_2(\omega)} h_2(\vartheta, \zeta^*(\vartheta))d\vartheta\right)\right)d\omega \tag{3.8}$$

where $\Upsilon^{*-1} = \frac{1}{1+\sum_{k=1}^{m} {a_k}^*} \neq 0$, such that $|a_k - a_k^*| < \delta$. Then

$$|\zeta(\eta)-\zeta^*(\eta)| = |\frac{1}{1+\sum_{k=1}^{m}a_k}\left[\zeta_0-\sum_{k=1}^{m}a_k\int_0^{\tau_k}\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right)\right.\right.$$

$$\left.\left.,\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right]$$

$$+\int_0^{\eta}\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega$$

$$-\frac{1}{1+\sum_{k=1}^{m}a_k{}^*}\left[\zeta_0^*-\sum_{k=1}^{m}a_k^*\int_0^{\tau_k}\varphi\left(\omega,\zeta^*\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta^*(\vartheta))d\vartheta\right),\zeta^*\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta^*(\vartheta))d\vartheta\right)\right)d\omega\right]$$

$$-\int_0^{\eta}\varphi\left(\omega,\zeta^*\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta^*(\vartheta))d\vartheta\right),\zeta^*\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta^*(\vartheta))d\vartheta\right)\right)d\omega|$$

$$\le\left|\frac{\zeta_0}{1+\sum_{k=1}^{m}a_k}-\frac{\zeta_0^*}{1+\sum_{k=1}^{m}a_k{}^*}\right|$$

$$+\left|\frac{-1}{1+\sum_{k=1}^{m}a_k}\sum_{k=1}^{m}a_k\int_0^{\tau_k}\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right.$$

$$\left.+\frac{1}{1+\sum_{k=1}^{m}a_k{}^*}\sum_{k=1}^{m}a_k{}^*\int_0^{\tau_k}\varphi\left(\omega,\zeta^*\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta^*(\vartheta))d\vartheta\right),\zeta^*\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta^*(\vartheta))d\vartheta\right)\right)d\omega\right|$$

$$+\int_0^{\eta}\left|\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)\right.$$

$$\left.-\varphi\left(\omega,\zeta^*\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta^*(\vartheta))d\vartheta\right),\zeta^*\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta^*(\vartheta))d\vartheta\right)\right)\right|d\omega$$

$$\le\Upsilon^{-1}\Upsilon^{*-1}\left[\left|\zeta_0-\zeta_0^*\right|+\left|\zeta_0\sum_{k=1}^{m}a_k^*-\zeta_0^*\sum_{k=1}^{m}a_k\right|\right]$$

$$+\left|\frac{-1}{1+\sum_{k=1}^{m}a_k}\sum_{k=1}^{m}a_k\int_0^{\tau_k}\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right.$$

$$\left.+\frac{1}{1+\sum_{k=1}^{m}a_k{}^*}\sum_{k=1}^{m}a_k\int_0^{\tau_k}\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right|$$

$$+\left|\frac{-1}{1+\sum_{k=1}^{m}a_k{}^*}\sum_{k=1}^{m}a_k\int_0^{\tau_k}\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right.$$

$$\left.+\frac{1}{1+\sum_{k=1}^{m}a_k{}^*}\sum_{k=1}^{m}a_k{}^*\int_0^{\tau_k}\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)\right|d\omega$$

$$+\left|\frac{-1}{1+\sum_{k=1}^{m}a_k{}^*}\sum_{k=1}^{m}a_k{}^*\int_0^{\tau_k}\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)d\omega\right.$$

$$\left.+\frac{1}{1+\sum_{k=1}^{m}a_k{}^*}\sum_{k=1}^{m}a_k{}^*\int_0^{\tau_k}\varphi\left(\omega,\zeta^*\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta^*(\vartheta))d\vartheta\right),\zeta^*\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta^*(\vartheta))d\vartheta\right)\right)\right|d\omega$$

$$+\int_0^{\eta}\left|\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)\right.$$

$$\left.-\varphi\left(\omega,\zeta^*\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta^*(\vartheta))d\vartheta\right),\zeta^*\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta^*(\vartheta))d\vartheta\right)\right)\right|d\omega$$

$$\leq \Upsilon^{-1}\Upsilon^{*-1}\left[\left|\zeta_0-\zeta_0^*\right|+\left|\zeta_0\sum_{k=1}^{m}a_k^*-\zeta_0\sum_{k=1}^{m}a_k+\zeta_0\sum_{k=1}^{m}a_k-\zeta_0^*\sum_{k=1}^{m}a_k\right|\right]$$

$$+\left|\frac{1}{1+\sum_{k=1}^{m}a_k^*}-\frac{1}{1+\sum_{k=1}^{m}a_k}\right|\sum_{k=1}^{m}a_k\int_0^{\tau_k}\left|\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right)\right.\right.$$

$$\left.\left.,\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)\right|d\omega$$

$$+\frac{1}{1+\sum_{k=1}^{m}a_k^*}\sum_{k=1}^{m}\left|a_k^*-a_k\right|\int_0^{\tau_k}\left|\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)\right|d\omega$$

$$+\frac{1}{1+\sum_{k=1}^{m}a_k^*}\sum_{k=1}^{m}a_k^*\int_0^{\tau_k}\left|\varphi\left(\omega,\zeta^*\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta^*(\vartheta))d\vartheta\right),\zeta^*\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta^*(\vartheta))d\vartheta\right)\right)\right.$$

$$\left.-\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)\right|d\omega$$

$$+\int_0^{\eta}\left|\varphi\left(\omega,\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right),\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right)\right.$$

$$\left.-\varphi\left(\omega,\zeta^*\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta^*(\vartheta))d\vartheta\right),\zeta^*\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta^*(\vartheta))d\vartheta\right)\right)\right|d\omega$$

$$\leq \Upsilon^{-1}\Upsilon^{*-1}\left[\delta_2+\left|\zeta_0\right|\sum_{k=1}^{m}\left|a_k^*-a_k\right|+\sum_{k=1}^{m}a_k\left|\zeta_0-\zeta_0^*\right|\right]$$

$$+\Upsilon^{-1}\Upsilon^{*-1}\sum_{k=1}^{m}\left|a_k-a_k^*\right|\sum_{k=1}^{m}a_k\int_0^{\tau_k}\left[\left|\varphi(\omega,0,0)\right|+b\left|\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right)\right|\right.$$

$$\left.+b\left|\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right|\right]d\omega$$

$$+\Upsilon^{*-1}\delta_1 m\int_0^{\tau_k}\left[\left|\varphi(\omega,0,0)\right|+b\left|\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right)\right|+b\left|\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right|\right]d\omega$$

$$+b\left(1-\Upsilon^{*-1}\right)\int_0^{\tau_k}\left|\zeta^*\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta^*(\vartheta))d\vartheta\right)-\zeta^*\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right)\right|d\omega$$

$$+b\left(1-\Upsilon^{*-1}\right)\int_0^{\tau_k}\left|\zeta^*\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right)-\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right)\right|d\omega$$

$$+b\left(1-\Upsilon^{*-1}\right)\int_0^{\tau_k}\left|\zeta^*\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta^*(\vartheta))d\vartheta\right)-\zeta^*\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right|d\omega$$

$$+b\left(1-\Upsilon^{*-1}\right)\int_0^{\tau_k}\left|\zeta^*\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)-\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)\right|d\omega$$

$$+b\int_0^\eta\left|\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right)-\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta^*(\vartheta))d\vartheta\right)\right|d\omega$$
$$+b\int_0^\eta\left|\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta^*(\vartheta))d\vartheta\right)-\zeta^*\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta^*(\vartheta))d\vartheta\right)\right|d\omega$$
$$+b\int_0^\eta\left|\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)-\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta^*(\vartheta))d\vartheta\right)\right|d\omega$$
$$+b\int_0^\eta\left|\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta^*(\vartheta))d\vartheta\right)-\zeta^*\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta^*(\vartheta))d\vartheta\right)\right|d\omega$$
$$\leq \Upsilon^{-1}\Upsilon^{*-1}\left[\delta_2|\zeta_0|\delta_1 m+\delta_2\sum_{k=1}^m a_k\right]$$
$$+\delta_1 mE^{-1}\Upsilon^{*-1}\sum_{k=1}^m a_k\int_0^{\tau_k}\left[|\varphi(\omega,0,0)|+b\left|\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right)-\zeta(0)\right|\right]$$
$$+b\left|\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)-\zeta(0)\right|+2b\left|\zeta(0)\right|\Bigg]d\omega$$
$$+\Upsilon^{*-1}\delta_1 m\int_0^{\tau_k}\left[|\varphi(\omega,0,0)|+b\left|\zeta\left(\int_0^{\phi_1(\omega)}h_1(\vartheta,\zeta(\vartheta))d\vartheta\right)-\zeta(0)\right|+b\left|\zeta\left(\int_0^{\phi_2(\omega)}h_2(\vartheta,\zeta(\vartheta))d\vartheta\right)-\zeta(0)\right|\right.$$
$$\left.+2b|\zeta(0)|\right]d\omega$$
$$+bL\left(1-\Upsilon^{*-1}\right)\int_0^{\tau_k}\int_0^{\phi_1(\omega)}|h_1(\vartheta,\zeta^*(\vartheta))-h_1(\vartheta,\zeta(\vartheta))|d\vartheta d\omega$$
$$+bL\left(1-\Upsilon^{*-1}\right)\int_0^{\tau_k}\int_0^{\phi_2(\omega)}|h_2(\vartheta,\zeta^*(\vartheta))-h_2(\vartheta,\zeta(\vartheta))|d\vartheta d\omega+2bT\|\zeta-\zeta^*\|\left(1-\Upsilon^{*-1}\right)$$
$$+bL\int_0^\eta\int_0^{\phi_1(\omega)}|h_1(\vartheta,\zeta(\vartheta))-h_1(\vartheta,\zeta^*(\vartheta))|d\vartheta d\omega$$
$$+bL\int_0^\eta\int_0^{\phi_2(\omega)}|h_2(\vartheta,\zeta(\vartheta))-h_2(\vartheta,\zeta^*(\vartheta))|d\vartheta d\omega+2bT\|\zeta-\zeta^*\|$$

$$\leq \Upsilon^{-1}\Upsilon^{*-1}\delta_1\delta_2 m|\zeta_0|+\Upsilon^{*-1}\delta_2-\Upsilon^{-1}\Upsilon^{*-1}\delta_2$$
$$+\delta_1 m\Upsilon^{*-1}\left(1-\Upsilon^{-1}\right)\int_0^{\tau_k}\left[|\varphi(\omega,0,0)|+bL\int_0^{\phi_1(\omega)}|h_1(\vartheta,\zeta(\vartheta))|d\vartheta\right.$$
$$\left.+bL\int_0^{\phi_2(\omega)}|h_2(\vartheta,\zeta(\vartheta))|d\vartheta+2b\left|\zeta(0)\right|\right]d\omega$$
$$+\Upsilon^{*-1}\delta_1 m\int_0^{\tau_k}\left[|\varphi(\omega,0,0)|+bL\int_0^{\phi_1(\omega)}|h_1(\vartheta,\zeta(\vartheta))|d\vartheta+bL\int_0^{\phi_2(\omega)}|h_2(\vartheta,\zeta(\vartheta))|d\vartheta\right.$$
$$\left.+2b|\zeta(0)|\right]d\omega$$
$$+bLb_1\left(1-\Upsilon^{*-1}\right)\int_0^{\tau_k}\int_0^{\phi_1(\omega)}|\zeta^*(\vartheta)-\zeta(\vartheta)|d\vartheta d\omega$$
$$+bLb_2\left(1-\Upsilon^{*-1}\right)\int_0^{\tau_k}\int_0^{\phi_2(\omega)}|\zeta^*(\vartheta)-\zeta(\vartheta)|d\vartheta d\omega+2bT\|\zeta-\zeta^*\|\left(1-\Upsilon^{*-1}\right)$$
$$+bLb_1\int_0^\eta\int_0^{\phi_1(\omega)}|\zeta(\vartheta)-\zeta^*(\vartheta)|d\vartheta d\omega+bLb_2\int_0^\eta\int_0^{\phi_2(\omega)}|\zeta(\vartheta)-\zeta^*(\vartheta)|d\vartheta d\omega$$
$$+2bT\|\zeta-\zeta^*\|$$
$$\leq \Upsilon^{-1}\Upsilon^{*-1}\delta_1\delta_2 m|\zeta_0|+\Upsilon^{*-1}\delta_2-\Upsilon^{-1}\Upsilon^{*-1}\delta_2+\delta mE^*\left(M_3+2bLT^2+2b|\zeta(0)|T\right)\left(2-\Upsilon^{-1}\right)$$
$$+bLb_1T^2\|\zeta-\zeta^*\|\left(2-\Upsilon^{*-1}\right)+bLb_2T^2\|\zeta-\zeta^*\|\left(2-\Upsilon^{*-1}\right)+2bT\|\zeta-\zeta^*\|\left(2-\Upsilon^{*-1}\right)$$

Hence:

$$\|\zeta-\zeta^*\| \le \frac{\Upsilon^{-1}{\Upsilon^*}^{-1}\delta_1\delta_2 m|\zeta_0| + {\Upsilon^*}^{-1}\delta_2 - \Upsilon^{-1}{\Upsilon^*}^{-1}\delta_2 + \delta m E^*\left(M_3+2bLT^2+2bNT\right)(2-\Upsilon^{-1})}{1-bT(2-{\Upsilon^*}^{-1})(b_1LT+b_2LT+2)} = \varepsilon.$$

Then the solution of the integral equation (3.1) depends continuously on the nonlocal parameter a_k and the initial data ζ_0.

Now, we present an example to illustrate the main results.

Example 3.1. Consider the differential equation

$$\frac{d\zeta}{d\eta} = \frac{1}{2}(\eta+5) + \frac{2}{(\eta-4)^2}\left[\zeta\left(\int_0^{\left(\frac{1}{3}\right)^{B\eta}} \frac{\sin\frac{\pi}{2}u}{1+\sqrt{|\zeta(u)|}}\right) + \zeta\left(\int_0^{\cos B\eta} \frac{\tan\frac{\pi}{4}u}{1+\ln|\zeta(u)|}\right)\right], \quad \eta, B \in (0,1], \tag{3.9}$$

with nonlocal condition

$$\zeta(0) + \sum_{k=1}^{m} k^3 \zeta\left(\frac{\ln k}{k}\right) = 1. \tag{3.10}$$

$$\begin{aligned}
\zeta(\eta) &= \frac{1}{1+\sum_{k=1}^{m}k^3}\left[1-\sum_{k=1}^{m}k^3\int_0^{\frac{\ln k}{k}}\left(\frac{1}{2}(\omega+5)+\frac{2}{(\omega-4)^2}\left(\zeta\left(\int_0^{\left(\frac{1}{3}\right)^{Bs}}\frac{\sin\frac{\pi}{2}u}{1+\sqrt{|\zeta(u)|}}du\right)\right.\right.\right.\\
&\quad \left.\left.\left.+\zeta\left(\int_0^{\cos Bs}\frac{\tan\frac{\pi}{4}u}{1+\ln|\zeta(u)|}du\right)\right)\right)\mathrm{d}\omega\right]\\
&\quad +\int_0^{\eta}\left(\frac{1}{2}(\omega+5)+\frac{2}{(\omega-4)^2}\left(\zeta\left(\int_0^{\left(\frac{1}{3}\right)^{B\omega}}\frac{\sin\frac{\pi}{2}u}{1+\sqrt{|\zeta(u)|}}du\right)\right.\right.\\
&\quad \left.\left.+\zeta\left(\int_0^{\cos B\omega}\frac{\tan\frac{\pi}{4}u}{1+\ln|\zeta(u)|}du\right)\right)\right)\mathrm{d}\omega\Big]
\end{aligned}$$

Set

$$\varphi\left(\eta,\zeta\left(\int_0^{\phi_1(\eta)} h_1(u,\zeta(u))\mathrm{d}u\right),\zeta\left(\int_0^{\phi_2(\eta)} h_2(u,\zeta(u))\mathrm{d}u\right)\right) =$$

$$\frac{1}{2}(\eta+5)+\frac{2}{(\eta-4)^2}\left(\zeta\left(\int_0^{\left(\frac{1}{3}\right)^{B\eta}} \frac{\sin\frac{\pi}{2}u}{1+\sqrt{|\zeta(u)|}}\right)+\zeta\left(\int_0^{\cos Bt}\frac{\pi}{4}u1+\ln\left|\zeta(u)\right|\right)\right).$$

Then

$$\varphi\left(\eta,\zeta\left(\int_0^{\phi_1(\eta)} h_1(u,\zeta(u))\mathrm{d}u\right),\zeta\left(\int_0^{\phi_2(\eta)} h_2(u,\zeta(u))\mathrm{d}u\right)\right)$$
$$\leq\frac{1}{2}(\eta+5)+\frac{2}{9}(|\zeta(h_1)|+|\zeta(h_2)|)$$

and also

$$|h_{2,3}(u,\zeta(u))|\leq 1.$$

It is clear that the assumptions 1–4 of Theorem 3.4 are satisfied with $|c(\eta)|=|\frac{1}{2}(\eta+5)|\leq 3$ is measurable bounded, $b=\frac{2}{9}$. Therefore, by applying to Theorem 3.4, the given nonlocal problem (3.8)-(3.9) has a solution given by the integral solution (3.11).

4. Conclusion

Firstly we introduced the self-reference functional integro-differential equation with nonlocal condition (1.1), and also the nonlocal problem of functional integro-differential equation with the infinite-point boundary condition (1.2).

Then we began our work by obtaining the integral equation (3.1) equivalent to (1.1). Then to prove the existence of at least one solution for both (3.1)–(1.2), we used Schauder fixed point theorem.

Then to prove the uniqueness of the solution of (3.1), we used Banach fixed point theorem. Then we proved the continuous dependence on the nonlocal parameter a_k and the initial data ζ_0 of equation (3.1). At the end, we presented an example to illustrate the main idea.

Ethics information

Ethics approval and consent to participate are not applicable.

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Prof. Dr. Adel Abd El-Fattah Darwish: Supervision, Conceptualization, Methods, Reviewing, and Editing.
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Prof. Dr. Hossam Hasan Abd El-Ghany: Supervision, Conceptualization, Methods, Reviewing, and Editing.
Dr. Reda Gamal Ahmed: Supervision, Conceptualization, Methods, Reviewing, and Editing.
Associate Prof. Dr. Ahmed Abd El-Monem El-Deeb: Supervision, Conceptualization, Methods, Reviewing, and Editing.
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Conflicts of interest

There is no conflict of interest regarding the publication of this paper.

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